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A CONVEX APPROXIMANT METHOD FOR NON-CONVEX EXTENSIONS OF GEOMETRIC PROGRAMMING

by

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1. Introduction:

Many important problems of engineering and management are of a form which could be represented as geometric programs except that the functional to be minimized as well as the constraints are not confined to "posynomials" [1/] in that some of the coefficients are negative. The resulting problem thus may not, in general, be transformed to an equivalent convex programming problem. [2/] To date the only general method for obtaining global optima to (necessarily non-convex) problems with multiple local optima is Gomory's integer programming method. [3/]

We are herewith proposing an approximate method for another class of problems with multiple local optima--*viz.*, extensions of geometric programming in which some of the coefficients are negative. This method provides, at each stage, a convex approximant which, *a fortiori*, provides the duality relations that are needed for many purposes. This is in contrast to other approaches which either lose these duality relations [4/] or else restrict the applications to special situations. [5/] More specifically,

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1/ Cf. [7] for definitions of this and other terminology in geometric programming.

2/ Cf., e.g., the exponential transformations used in [3] and [4].

3/ See [8] and [9] for Gomory's original articles. See also [2] and [6] for further discussion and development.

4/ Cf., e.g., [10].

5/ The constraints in [3] and [5] , for instance, were arranged so that they could always be treated in a manner which did not preclude access to the indicated duality. Other possibilities are also present, however, as witness some of the examples, treated in [7].

the method that we shall describe here is conceived in the same spirit as previous suggestions we have made as a result of other research we have conducted to extend the boundaries of ordinary linear programming.[1/]

2. Formulation and Development of the Convex Approximant:

Consider the following problem[2/]

$$\text{(1.3)} \qquad \begin{array}{ll} \min & g_o^+ - g_o^- \\ \text{subject to} & \\ & g_i^+ - g_i^- \leq 1, \; i=1, \ldots, m \end{array}$$

where the g_k^+, g_k^- are posynomials in

$$\text{(1.2)} \qquad t = (t_1, \ldots, t_n) \; .$$

I.e.,

$$g_i^+ = \sum_{j \epsilon J_i} P_{ij}^+ (t) \; ; \quad g_i^- = \sum_{k \epsilon K_i} P_{ij}^-(t)$$

$$\text{(1.3)} \qquad \begin{array}{l} P_{ij}^+(t) = c_{ij}^+ \, t_1^{a_1^{ij}} \; \cdots \; t_n^{a_n^{ij}} \\ P_{ij}^-(t) = c_{ij}^- \, t_1^{b_1^{ij}} \; \cdots \; t_n^{b_n^{ij}} \end{array}$$

$$c_{ij}^+, \, c_{ij}^- > 0 \; .$$

1/ Cf., e.g., [1] and [5].

2/ To abbreviate this part of the development, it is assumed that all conditions for existence and attainment of the indicated minima are fulfilled. Cf. [7] for a rigorous treatment of the relevant necessary and sufficient conditions in complete detail.

Note that the above problem is a generalization of ordinary geometric programming in that the constraints and the functional are not confined to posynomials.

3. Formulation of Approximants:

Each one-term posynomial $\bar{P}_{ij}(t)$ in the preceding expressions may be replaced by a single variable y_{ij} subject to

$$y_{ij} \leq \bar{P}_{ij}(t) \tag{2.1}$$

or

$$y_{ij} [\bar{P}_{ij}(t)]^{-1} \leq 1 \tag{2.2}$$

which is the same as

$$\frac{y_{ij}}{\bar{c}_{ij}} \left[t_1^{-b_1^{ij}} \cdots t_n^{-b_n^{ij}} \right] \leq 1 . \tag{2.3}$$

The resulting problem in t and the y_{ij} is equivalent to (1.1).

Next, let us suppose that the range of each y_{ij} relevant to the optimization may be represented by

$$0 < L_{ij} \leq y_{ij} \leq U_{ij} \tag{3}$$

We then introduce $k_{ij} > U_{ij}$ and consider the function

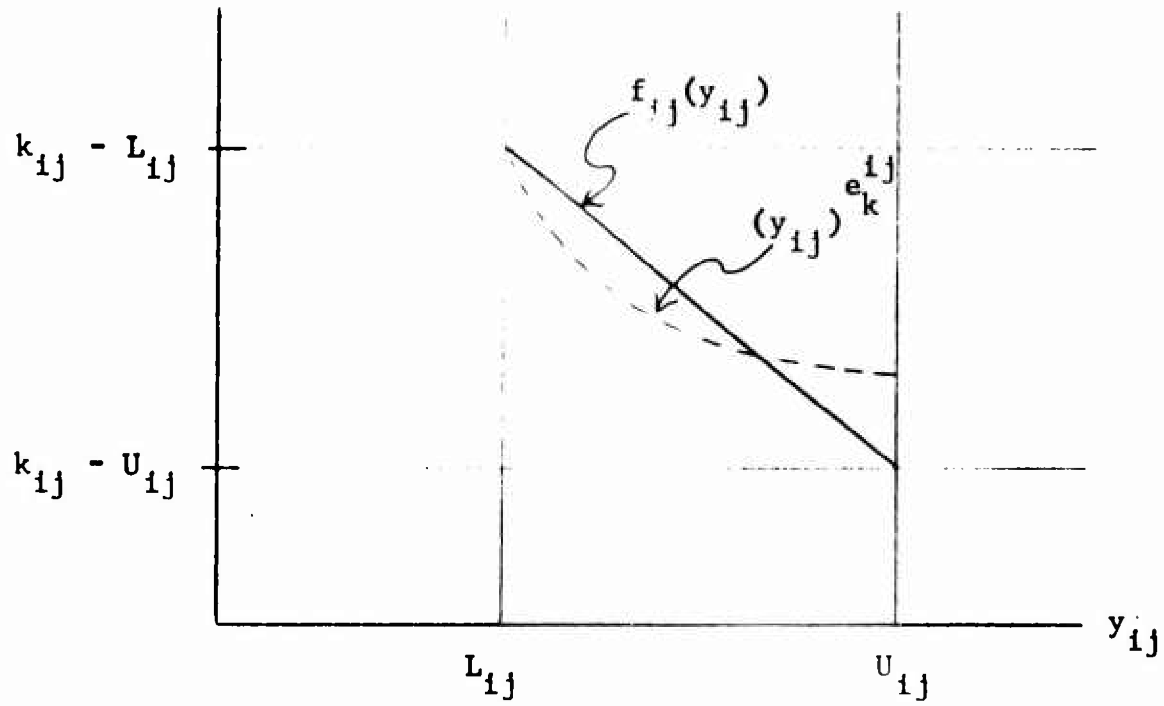
$$f_{ij}(y_{ij}) = k_{ij} - y_{ij} \tag{4}$$

as diagrammed below. Evidently over the interval (L_{ij}, U_{ij}) the linear function (4) is positive and bounded above and below. It may thus be

approximated by a posynomial

(5) $$q_{ij}(y_{ij}) = \sum_k d_k^{ij} (y_{ij})^{e_k^{ij}}$$

where the d_k^{ij} are suitably selected positive constants.



To the degree of approximation thus rendered--e.g., approximation of the linear function by posynomials--the original problem (1.1) is now replaced by

$$\min \quad g_o^+(t) + \sum_{j=1}^{m_o} q_{oj}\,(y_{oj})$$

subject to

$$g_i^+(t) + \sum_{j=1}^{m_o} q_{ij}\,(y_{ij}) \leq 1 + \sum_j k_{ij} \tag{6}$$

$$[P_{ij}^-(t)]^{-1}\, y_{ij} \leq 1$$

$$y_{ij}\, U_{ij}^{-1} \leq 1$$

$$y_{ij}^{-1}\, L_{ij} \leq 1$$

$$t > 0$$

This problem may evidently be transformed (e.g., by the exponential transformation)[1/] into a convex programming problem. We therefore call it a convex approximant of the original problem. It therefore follows that it has only one local (= global) optimum value.

Note in particular that each convex approximant has an associated dual problem. Thus a dual evaluator is available for each constraint. Those that refer to the U_{ij}, L_{ij} constraints indicate possible directions of improvement if these upper or lower bounds are tight. The dual evaluator is, of course, equal to zero when these bounds are slack. The approximation can thus be improved in the neighborhood of any already attained optimum by, e.g., reducing the range of the slack U_{ij} and L_{ij}, thereby enabling one to make an improved posynomial fit in the next

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1/ See [3] and [4].

convex programming approximant. Similarly, the interval may be reduced and translated in the direction indicated by the non-zero dual evaluator for the tight U_{ij}, L_{ij} constraints.

Thus, sequentially, the convex approximant can be refined. One would expect the global optimum to be obtained by this method in situations where the original problem has multiple local optima. For, if the global optimum value were significantly different from that of other local optima, one would anticipate that the small modifications of the smooth continuous functions to equally smooth continuous approximants would not significantly alter the global optimum. Since the convex approximant has only one local (= global) optimum, its value should therefore be close to the global optimum value of the original problem. On the other hand, when the global optimum value of the original does not differ significantly from other local optimum values, the precise optimum obtained matters little so far as value is concerned. In either situation therefore one would expect a sequence of convex approximants to yield a worthwhile result.

3. Conclusion:

In the paper [4], we showed how geometric programming could be applied to the determination of multiple simultaneous EOQ (economic order quantity) formulas under constraints as well as to aspects of the economic theory of production (e.g., with Cobb-Douglas and generalized SMAC production functions). Still further extensions in this direction

(e.g., to problems of capital budgeting) critically depend on the possibility of dealing with the presence of negative coefficients--as in (1.1)--and the same is true even of the originally motivated applications to engineering designs when, for instance, scrap values require consideration. Even more important, however, is the need for increased flexibility as when, for instance, there is a need to deal with problems where the natural original orientation is toward maximization (rather than minimization) and where a restriction to posynomials only makes it impossible to proceed through the negative of an associated minimization problem.[1/] A recourse to the convex approximant method would then seem to be in order--at least in these cases and possibly others as well.

1/ E.g., as in ordinary linear programming. Cf., e.g., [2] or [6].

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13. ABSTRACT

Many important problems of engineering and management are of a form which could be represented as geometric programs except that the functional to be minimized as well as the constraints are not confined to posynomials in that some of the coefficients are negative. This paper supplies a way for dealing with such negative terms by a constraint adjunction procedure which yields an associated approximating problem involving only posynomials which can, in turn, be transformed into a convex programming problem that has only one local (= global) optimum. The latter, which is called a convex approximant, has an associated dual. Recourse to the related duality theory then supplies guidance for improving the approximation along lines that are indicated in the paper.

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